PERISTALTIC FLOW OF A VISCOUS INCOMPRESSIBLE FLUID IN A PLANE CHANNEL

V. I. Vishnyakov, K. B. Pavlov,
and V. I. Semenov

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This article shows the effect of additional pulsations of liquid flow rate on peristaltic flow in a channel and the formation of zones of eddy flow in it.

In connection with the rapid growth of biomechanics, the study of questions relating to different regimes of peristaltic pumping is of great interest. In such pumping, liquid is transferred in a channel as a result of the propagation of a traveling wave along its elastic walls. Features of the hydrodynamics involved have been examined in detail in both planar and cylindrical channels [1-7]. However, in biological systems one encounters regimes in which the traveling wave on the elastic walls of the channel is propagated in a direction which does not coincide with the direction of the main flow. This occurs, for example, in the flow of blood in arterioles [8]. In connection with this, we examined the peristaltic flow of an incompressible viscous liquid in a plane channel with additional pumping (liquid flow rate) which changed periodically over time due to an external pressure gradient.

Let the deformation of the elastic walls of the channel be described by the equation of a traveling wave $Y = F(X - Wt)$ with a length Λ much greater than the mean half-width of the channel H; we also assume that the flow rate of the liquid in the channel changes periodically with a frequency Ω.

It would be best to further conduct our investigation in a reference frame which moves along the channel axis with the phase velocity of the traveling wave W; in this reference frame, the deformation of the elastic walls of the channel is fixed and is independent of time: $Y = F(X)$.

As the characteristic quantities we take the half-width of the channel H and the mean half-flow-rate of the liquid Q, determined by the relation:

$$Q = \frac{1}{T}\int_0^T \int_0^{F(x,\,t)} \frac{\partial \Psi}{\partial Y}\, dY dt, \text{ where } T = 2\pi/\Omega;\ \Psi = \Psi(X, Y, t)$$

is the stream function. Then the motion of the liquid can be characterized by the following dimensionless parameters: Reynolds number $\mathrm{Re} = Q/\nu$; dimensionless phase velocity $w = WH/Q$.

Here, the dimensionless equation of motion, written in the "moving" reference frame relative to the stream function $\psi(x, y, t)$, has the form

$$\frac{\partial}{\partial t}\Delta\psi + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\Delta\psi - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\Delta\psi = \frac{1}{\mathrm{Re}}\Delta\Delta\psi. \tag{1}$$

We will assume that in a stationary reference frame the flow rate of the liquid can be represented as the sum of two terms: a constant term and a term dependent on time in accordance with a harmonic law and having the frequency Ω. Then the dimensionless stream function should satisfy the following boundary conditions in the "moving" reference frame on the elastic walls of the channel:

$$\psi|_{y=F(x)} = 1 - w + q\cos\omega t,\quad \left.\frac{\partial \psi}{\partial y}\right|_{y=F(x)} = -w. \tag{2}$$

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Here q = const; $\omega = \Omega H/W$ is the dimensionless frequency. The conditions imposed on the stream function in the center of the channel y = 0 have the form

$$\psi|_{y=0} = \left.\frac{\partial^2\psi}{\partial y^2}\right|_{y=0} = 0. \tag{3}$$

Since $\Lambda \gg H$, then boundary-value problem (1-3) can be solved by the asymptotic "narrow-band" method [9]. We seek solutions of (1) in the form

$$\psi = \sum_{i=0}^{\infty} \varepsilon^i \psi_i \quad (\varepsilon = H/\Lambda). \tag{4}$$

Limiting ourselves to just the first two terms, we obtain the system of equations

$$\frac{\partial^3\psi_0}{\partial t\partial y^2} = \frac{1}{\mathrm{Re}}\frac{\partial^4\psi_0}{\partial y^4}, \tag{5}$$

$$\frac{\partial^3\psi_1}{\partial t\partial y^2} = \frac{1}{\mathrm{Re}}\frac{\partial^4\psi_1}{\partial y^4} + \frac{\partial\psi_0}{\partial x}\frac{\partial^3\psi_0}{\partial y^3} - \frac{\partial\psi_0}{\partial y}\frac{\partial^3\psi_0}{\partial x\partial y^2}. \tag{6}$$

The boundary conditions (2) and (3) for the corresponding approximations are written in the form

$$\psi_0|_{y=F(x)} = 1 - w + q\cos\omega t, \quad \left.\frac{\partial\psi_0}{\partial y}\right|_{y=F(x)} = -w,$$

$$\psi_0|_{y=0} = \left.\frac{\partial^2\psi_0}{\partial y^2}\right|_{y=0} = 0, \tag{7}$$

$$\psi_1|_{y=F(x)} = \left.\frac{\partial\psi_1}{\partial y}\right|_{y=F(x)} = \psi_1|_{y=0} = \left.\frac{\partial^2\psi_1}{\partial y^2}\right|_{y=0} = 0. \tag{8}$$

We seek the real solution of system (5)-(6) by means of series written in complex form:

$$\psi_0(x, y, t) = \sum_{n=-\infty}^{n=\infty} \alpha_n(x, y)\exp(in\omega t), \tag{9}$$

$$\psi_1(x, y, t) = \sum_{n=-\infty}^{n=\infty} \beta_n(x, y)\exp(in\omega t), \tag{10}$$

where the coefficients $\alpha_n(x, y)$ and $\beta_n(x, y)$ should satisfy the condition

$$\alpha_n = \alpha^*_{-n}, \quad \beta_n = \beta^*_{-n}. \tag{11}$$

Here, the asterisk denotes complex-conjugate quantities.

Using Eqs. (9) and (11) and boundary conditions (7) and separating the real and imaginary parts, we obtain an expression for the stream function in the zeroth approximation

$$\psi_0(x, y, t) = \alpha_0 + 2[\mathrm{Real}\,\alpha_1\cos\omega t - \mathrm{Im}\,\alpha_1\sin\omega t],$$

$$\alpha_0 = -\frac{\xi(x)}{6}\mathrm{Re}\,y^3 + \varkappa(x)y, \quad \xi(x) = \frac{3}{\mathrm{Re}\,F^3}(1 - w + wF),$$

$$\varkappa(x) = \frac{\xi(x)\,\mathrm{Re}}{2}F^2 - w, \quad \varphi = \sqrt{\frac{\omega\,\mathrm{Re}}{2}},$$

$$\mathrm{Real}\,\alpha_1 = \frac{\eta(x)}{\omega}y + 2\vartheta_1(x)\,\mathrm{sh}(\varphi y)\cos(\varphi y) - 2\vartheta_2(x)\,\mathrm{ch}(\varphi y),$$

$$\mathrm{Im}\,\alpha_1 = \frac{\eta(x)}{\omega}y + 2\vartheta_1(x)\,\mathrm{ch}(\varphi y)\sin(\varphi y) + 2\vartheta_2(x)\,\mathrm{sh}(\varphi y)\cos(\varphi y),$$

$$\eta(x) = 2\varphi\omega\delta(x)[\vartheta_1(x) + \vartheta_2(x)],$$

$$\delta(x) = \mathrm{sh}(\varphi F)\sin(\varphi F) + \mathrm{ch}(\varphi F)\cos(\varphi F),$$

$$\vartheta_1(x) = \vartheta_2(x)\frac{\mu_1(x)}{\mu_2(x)}, \quad \vartheta_2(x) = \frac{q}{2}\,\frac{\mu_1(x)}{\mu_1^2(x) + \mu_2^2(x)},$$

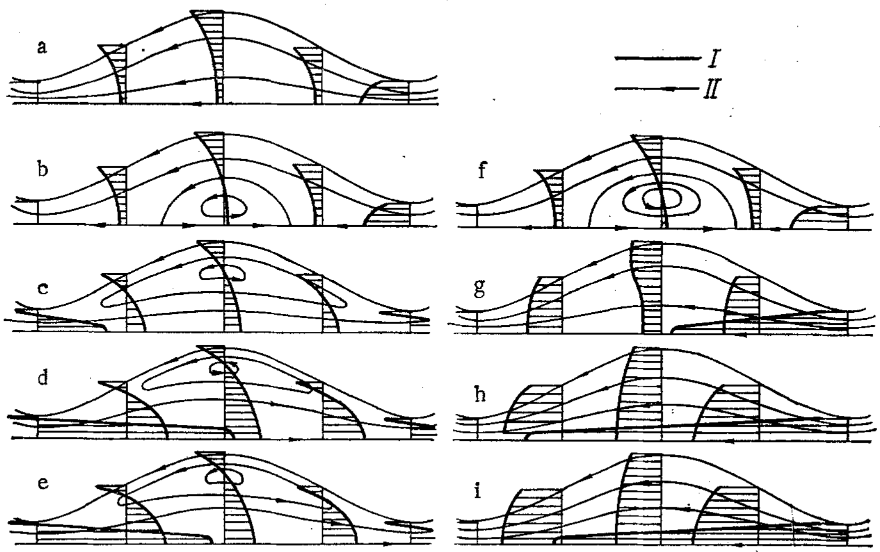

Fig. 1. Complete cycle of development of peristaltic flow with additional variable flow rate after nine successive moments of time (flow parameters: A = 0.77, w = 2, ω = 0.04, q = −2): I) velocity profiles; II) stream functions; a) t = 0; b) $\frac{1}{96}T$; c) $\frac{1}{12}T$; d) $\frac{3}{12}T$; e) $\frac{4}{12}T$; f) $\frac{6}{12}T$; g) $\frac{7}{12}T$; h) $\frac{9}{12}T$; i) $\frac{10}{12}T$.

$$\mu_1(x) = 2\varphi F\delta(x) - \operatorname{sh}(\varphi F)\cos(\varphi F),$$
$$\mu_2(x) = \mu_1(x) + \operatorname{sh}(\varphi F)\cos(\varphi F) - \operatorname{ch}(\varphi F)\sin(\varphi F).$$

We similarly find the stream function in the first approximation; however, its explicit expression is omitted here due to its awkwardness.

Thus, the asymptotic solution satisfies boundary-value problem (1)-(3) to within the terms $O(\varepsilon^2)$.

This solution was used to numerically model the flow on an ES 1033 computer, with the generation of graphs on an alphanumeric printer for the special case when the deformation of the elastic walls of the channel in the "moving" reference frame is described by the expression $F(x) = 1 + A\cos(2\pi x/\lambda)$ (A is the amplitude and $\lambda = \Lambda/H$ is the dimensionless length of the wave). To more clearly trace the effect of the governing parameters of the problem on the character of flow, we first modeled peristaltic flow in the liquid "capture" regime [2].

This phenomenon, characteristic of peristaltic flow, is shown in Fig. 1f with the governing parameters w = 2 and A = 0.77. Such parameters were chosen so that the prescribed constant flow rate Q would ensure only peristaltic pumping, without an additional axial pressure gradient.

With the addition of a variable gradient flow with a flow rate prescribed according to the law $q\cos\omega t$ to the peristaltic flow in the "capture" regime, a fairly complicated nonsteady peristaltic flow results. A characteristic pattern of such a flow is shown in Fig. 1 after nine successive moments of time for the case when q = 2 and ω = 0.04.

It can be seen from this figure that the entire period of development of the flow can be broken down into two phases: the first phase, when there is an eddy zone of liquid "capture"; the second phase, when there is no such zone. At the beginning of the first phase, an increase in flow rate is accompanied by intensification of the main peristaltic flow. The eddy zone grows and divides in two, moving away from the channel axis, stretching out, and rising under the wave crest. With a decrease in flow rate the eddy zone decreases in size, descends toward the channel axis, and disappears. In the second phase there are no

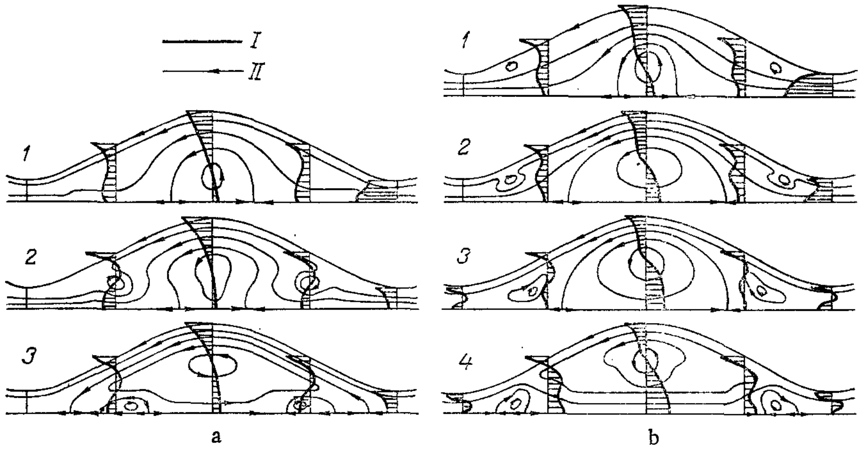

Fig. 2. Effect of frequency of pulsations of additional variable flow rate on the formation of "secondary" eddies: I) velocity profiles; II) stream functions; a) $\omega = 0.2$; 1) $t = \frac{4}{288}T$; 2) $\frac{5}{288}T$; 3) $\frac{6}{288}T$; b) $\omega = 1$; 1) $t = \frac{2}{288}T$; 2) $\frac{4}{288}T$; 3) $\frac{6}{288}T$; 4) $\frac{8}{488}T$.

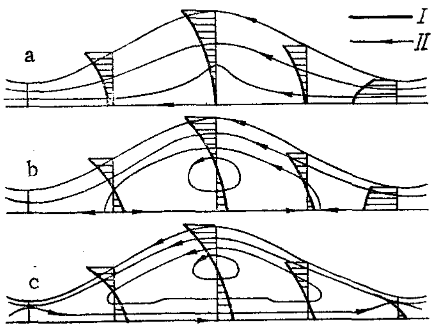

Fig. 3. Characteristic flow patterns at low frequencies of flow-rate pulsations ($A = 0.77$, $w = 2$, $q = -2$; $\omega = 0.04$); a) $t = \frac{2}{288}$ T; b) $\frac{4}{288}T$; c) $\frac{6}{288}T$.

eddy zones and the flow becomes similar to peristaltic flow when the "capture" regime is absent. The additional flow rate at this time is directed counter to the peristaltic flow.

More detailed study of the above flow regime revealed the existence of additional — "secondary" — eddy zones of flow near the bends in the channel wall (see Fig. 2).

Such eddy zones were not seen at the frequency $\omega = 0.04$ (see Fig. 3) but appear at higher frequencies of flow-rate pulsation. Meanwhile, the configuration of the eddy zone becomes more complicated with increasing frequency, as is well illustrated by the restructuring of the velocity profile. Thus, for example, the latter is restructured from a parabolic profile (Fig. 3) to an M-shaped profile (Fig. 2a) and an even more complicated shape (Fig. 2b).

The duration of the "secondary" eddies and the size of the region they occupy in the channel are less than for the main eddy flow. The "secondary" eddies move toward the channel axis, are enveloped by the main eddy, and disappear on the channel axis.

Thus, the frequency of the flow-rate pulsations exerts the main effect on the character of flow and the formation of eddy zones. An increase in frequency is accompanied by a sharp increase in the pressure gradient in the constricted parts of the channel, which must be considered in calculating corresponding flow regimes. The amplitudes of the pulsations have less of an effect. The flow regimes change from forward to reverse more quickly with an increase in amplitude, but no "secondary" eddies are formed.

NOTATION

X, Y, Cartesian coordinates; t, time; W, w, dimensional and dimensionless phase velocities, respectively; Λ; wavelength; A, wave amplitude; H, half-width of the channel; Q, mean half-flow-rate of the liquid; Ψ, ψ, dimensional and dimensionless stream functions, respectively; Re, Reynolds number; ν, kinematic viscosity of the medium; ε, small parameter; q, sought constant; $\alpha, \beta, \varkappa, \xi, \mu, \eta, \varphi, \vartheta$, coefficients; T, period of pulsations of additional variable flow rate.

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